Inverse Problem for a Curved Quantum Guide

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June 14, 2006

Abstract

In this paper, we consider the Dirichlet Laplacian operator on a curved quantum guide in $\mathbb{R}^n$ ($n = 2, 3$). If the reference curve is not straight but asymptotically straight, we prove that the data of one eigenfunction associated with its eigenvalue determines the curvature.

1 Introduction

This paper is organized as follows: in section 1, we define the quantum guide we are studied for in $\mathbb{R}^n$ ($n = 2, 3$) and we present our results. In section 2 (resp. section 3), we give the proofs of these results for $n = 2$ (resp. for $n = 3$). Finally in section 4, we give a generalization in $\mathbb{R}^n$ ($n \geq 2$).

1.1 Uniqueness result for a $\mathbb{R}^2$-quantum guide

We consider the Laplacian operator on a curved quantum guide $\Omega \subset \mathbb{R}^2$, with Dirichlet boundary conditions, denoted by $-\Delta_D^\Omega$. We proceed as in [4]. We denote by $\Gamma = (\Gamma_1, \Gamma_2)$ the function which characterizes the reference curve and by $N = (N_1, N_2)$ the...
outgoing normal. We denote by $d$ the fixed width of $\Omega$ and by $\Omega_0 := \mathbb{R} \times [-d/2, d/2]$. Each point $(x, y)$ of $\Omega$ is described by the curvilinear coordinates $(s, u)$ as follows:

$$f : \Omega_0 \rightarrow \Omega \text{ with } (x, y) = f(s, u) = \Gamma(s) + uN(s). \quad (1.1)$$

We assume $\Gamma'_1(s)^2 + \Gamma'_2(s)^2 = 1$ and we recall that the signed curvature $\gamma$ of $\Gamma$ is defined by:

$$\gamma(s) = -\Gamma''_1(s)\Gamma'_2(s) + \Gamma''_2(s)\Gamma'_1(s), \quad (1.2)$$

named so because $|\gamma(s)|$ represents the curvature of the reference curve at $s$. We recall that a guide is called simply-bent if $\gamma$ does not change sign in $\mathbb{R}$. We assume throughout Subsection 1.1 and Section 2 that:

**Assumption 1:** $\gamma \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\frac{d}{2} < \frac{1}{\|\gamma\|_\infty}$, $\gamma(s) \to 0$ as $|s| \to +\infty$.

Note that, by the inverse function theorem, the map $f$ (defined by (1.1)) is a diffeomorphism provided $1 - u\gamma(s) \neq 0$, for all $u, s$, which is guaranteed by Assumption 1. Note also that $1 - u\gamma(s) > 0$ for all $u$ and $s$. (More precisely, $0 < 1 - \frac{d}{2}\|\gamma\|_\infty \leq 1 - u\gamma(s) \leq 1 + \frac{d}{2}\|\gamma\|_\infty$ for all $u, s$.) The curvilinear coordinates $(s, u)$ are locally orthogonal, so the metric in $\Omega$ is expressed with respect to them through a diagonal metric tensor

$$(g_{ij}) = \begin{pmatrix} (1 - u\gamma(s))^2 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.3)$$

The transition to the curvilinear coordinates represents an isometric map of $L^2(\Omega)$ to $L^2(\Omega_0, g^{1/2} \, ds \, du)$ where

$$(g(s, u))^{1/2} := 1 - u\gamma(s) \quad (1.4)$$

is the Jacobian $\frac{\partial (x, y)}{\partial (s, u)}$. So we can replace the Laplacian operator $-\Delta^\Omega_D$ acting on $L^2(\Omega)$ by the Laplace-Beltrami operator $H_g$ acting on $L^2(\Omega_0, g^{1/2} \, ds \, du)$ relative to the given metric tensor $(g_{ij})$ (see (1.3) and (1.4)) where:

$$H_g := -g^{-1/2}\partial_s(g^{-1/2}\partial_s) - g^{-1/2}\partial_u(g^{1/2}\partial_u). \quad (1.5)$$

We rewrite $H_g$ (defined by (1.5)) into a Schrödinger-type operator acting on $L^2(\Omega_0, ds \, du)$. Indeed, using the unitary transformation

$$U_g : L^2(\Omega_0, g^{1/2} \, ds \, du) \xrightarrow{\psi} L^2(\Omega_0, ds \, du) \quad g^{1/4}\psi \quad (1.6)$$

setting

$$H_\gamma := U_g H_g U_g^{-1}, \quad (1.7)$$

we get

$$H_\gamma = -\partial_s(c_\gamma(s, u)\partial_s) - \partial^2_u + V_\gamma(s, u) \quad (1.8)$$

with

$$c_\gamma(s, u) = \frac{1}{(1 - u\gamma(s))^2} \quad (1.9)$$
and

\[ V_\gamma(s, u) = -\frac{\gamma^2(s)}{4(1 - u\gamma(s))^2} - \frac{u\gamma''(s)}{2(1 - u\gamma(s))^3} - \frac{5u^2\gamma'^2(s)}{4(1 - u\gamma(s))^4}. \]

(1.10)

**Remarks:** We will assume throughout all this subsection that \( \gamma^{(k)} \in L^\infty(\mathbb{R}) \) for each \( k = 0, 1, 2 \).

1. Then, note that such operator \( H_\gamma \) admits bound states and that the minimum eigenvalue \( \lambda_1 \) is simple and associated with a positive eigenfunction \( \phi_1 \) (see \cite[Sec.8.17]{5}). Furthermore, since \( \Omega \) is non trivially-curved and asymptotically straight, the operator \( -\Delta_\Omega^D \) has at least one eigenvalue of finite multiplicity below its essential spectrum (see \cite{2}; see also \cite{4} under the additional assumptions that the width \( d \) is sufficiently small and the curvature \( \gamma \) is rapidly decaying at infinity; see \cite{6} under the assumption that the curvature \( \gamma \) has a compact support).

2. Finally, note also that \( (\lambda, \phi) \) is an eigenpair (i.e. an eigenfunction associated with its eigenvalue) of the operator \( H_\gamma \) acting on \( L^2(\Omega_0, dsdu) \) means that \( (\lambda, U_g^{-1}\phi) \) is an eigenpair of \( -\Delta_\Omega^D \) acting on \( L^2(\Omega) \). (Recall that \( H_\gamma \) is defined by (1.7) and \( U_g \) by (1.6)). So the data of one eigenfunction of the operator \( H_\gamma \) is equivalent to the data of one eigenfunction of \( -\Delta_\Omega^D \).

The aim of this subsection is to prove that the data of one eigenpair determines uniquely the curvature. First, under

**Assumption 2:** \( \gamma \in C^5(\mathbb{R}) \) and \( \gamma^{(k)} \in L^\infty(\mathbb{R}) \) for each \( k = 0, \ldots, 5 \),

where \( \gamma^{(k)} \) denotes the \( k \)th derivatives of \( \gamma \), we obtain the following result:

**Theorem 1.1.** Let \( \Omega \) be the curved guide in \( \mathbb{R}^2 \) defined as above. Let \( \gamma \) be the signed curvature defined by (1.2) and satisfying Assumptions 1 and 2. Let \( H_\gamma \) be the operator defined by (1.8) and \( (\lambda, \phi) \) be an eigenpair of \( H_\gamma \). Then \( \gamma^2(s) = -4\frac{\Delta\phi(s,0)}{\phi(s,0)} - \lambda \) for all \( s \) when \( \phi(s,0) \neq 0 \).

**Remark:** Note that the condition \( \phi(s,0) \neq 0 \) imposed on \( \phi \) in Theorem 1.1 is satisfied for the positive eigenfunction \( \phi_1 \) and for all \( s \in \mathbb{R} \).

In the case of a simply-bent guide (i.e. when \( \gamma \) does not change sign in \( \mathbb{R} \)), we can restrain the hypotheses upon the regularity of \( \gamma \). Under

**Assumption 3:** \( \gamma \in C^3(\mathbb{R}) \) and \( \gamma^{(k)} \in L^\infty(\mathbb{R}) \) for each \( k = 0, \ldots, 3 \),

we obtain the following result:

**Theorem 1.2.** Let \( \Omega \) be the curved guide in \( \mathbb{R}^2 \) defined as above. Let \( \gamma \) be the signed curvature defined by (1.2) and satisfying Assumptions 1 and 3. We assume also that \( \gamma \) is a nonnegative function. Let \( H_\gamma \) be the operator defined by (1.8) and \( (\lambda, \phi) \) be an eigenpair of \( H_\gamma \). Then \( (\lambda, \phi) \) determines uniquely \( \gamma \).
Note that the above result is still valid for a nonpositive function $\gamma$. Finally, note also that under Assumption 4:

$$\gamma \in C^2(\mathbb{R}) \text{ and } \gamma^{(k)} \in L^\infty(\mathbb{R}) \text{ for each } k = 0, 1, 2$$

and still under condition of nonnegativity or nonpositivity of $\gamma$, we obtain the same result as in Theorem 1.2 for a curvature with a compact support.

### 1.2 Uniqueness result for a $\mathbb{R}^3$-quantum guide

Now, we apply the same ideas for a tube $\Omega$ in $\mathbb{R}^3$. We proceed here as in [2]. We give the notations which are valid throughout Subsection 1.2 and Section 3 (for the proofs). Let $s \mapsto \Gamma(s), \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$, be a curve in $\mathbb{R}^3$. We assume that $\Gamma$ is sufficiently smooth and possesses a positively oriented Frenet frame $\{e_1, e_2, e_3\}$. Given a bounded open connected neighborhood $\omega$ of $(0, 0) \in \mathbb{R}^2$, let $\Omega_0$ denote the straight tube $\mathbb{R} \times \omega$. We define the curved tube $\Omega$ of cross-section $\omega$ about $\Gamma$ by

$$\Omega := f(\mathbb{R} \times \omega) = f(\Omega_0), \quad f(s, u_2, u_3) := \Gamma(s) + R(s)[(u_2 + u_3)(e_2 + e_3)] \quad (1.11)$$

with $u = (u_2, u_3) \in \omega$ and

$$R(s) = \begin{pmatrix}
\cos(\theta(s)) & -\sin(\theta(s)) \\
\sin(\theta(s)) & \cos(\theta(s))
\end{pmatrix},$$

$\theta$ being a real-valued differentiable function such that $\theta'(s) = \tau(s)$ the torsion of $\Gamma$.

Note that $R$ is a rotation matrix in $\mathbb{R}^2$ chosen in such a way that $(s, u_2, u_3)$ are orthogonal "coordinates" in $\Omega$, i.e. $\omega$ rotating along $\Gamma$ w.r.t. the Tang frame.

Let $k$ be the first curvature function of $\Omega$. Recall that since $\Omega \subset \mathbb{R}^3$, $k$ is a nonnegative function. We assume throughout all this subsection that the following hypothesis holds:

**Assumption 5:**

i) $k \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad a := \sup_{u \in \omega} \|u\| < \frac{1}{\|k\|_\infty}, \quad k(s) \to 0 \text{ as } |s| \to +\infty$

ii) $\Omega$ does not overlap.

The Assumption 5 assures that the map $f$ (defined by (1.11)) is a diffeomorphism (see [2]) in order to identify $\Omega$ with the Riemannian manifold $(\Omega_0, (g_{ij}))$ where $(g_{ij})$ is the metric tensor induced by $f$, i.e. $(g_{ij}) := ^t J f J f$, ($J f$ denoting the Jacobian matrix of $f$). Recall that $(g_{ij}) = diag(h^2, 1, 1)$ (see [2]) with

$$h(s, u_2, u_3) := 1 - k(s)(\cos(\theta(s))u_2 + \sin(\theta(s))u_3). \quad (1.12)$$

Note that Assumption 5 implies that $0 < 1 - a\|k\|_\infty \leq 1 - h(s, u_2, u_3) \leq 1 + a\|k\|_\infty$ for all $s \in \mathbb{R}$ and $u = (u_2, u_3) \in \omega$. Moreover, setting

$$g := h^2 \quad (1.13)$$
we can replace the Dirichlet Laplacian operator \(-\Delta^D_\Omega\) acting on \(L^2(\Omega)\) by the Laplace-Beltrami operator \(K_g\) acting on \(L^2(\Omega_0, hdsdu)\) relative to the metric tensor \((g_{ij})\). We can rewrite \(K_g\) into a Schrödinger-type operator acting on \(L^2(\Omega^0_0, dsdu)\). Indeed, using the unitary transformation \(W_g: L^2(\Omega^0_0, hdsdu) \rightarrow L^2(\Omega^0_0, dsdu)\) defined by (1.14), we get

\[
W_g : L^2(\Omega_0, hdsdu) \quad \mapsto \quad L^2(\Omega_0, dsdu) \quad \mapsto \quad g^{1/4}\psi
\]

(1.14)

setting

\[
H_k := W_g K_g W_g^{-1},
\]

(1.15)

we get

\[
H_k = -\partial_s(h^{-2}\partial_s) - \partial^2_{u_2} - \partial^2_{u_3} + V_k
\]

(1.16)

where \(\partial_s\) denotes the derivative relative to \(s\) and \(\partial_{u_i}\) denotes the derivative relative to \(u_i\) and with

\[
V_k := -\frac{k^2}{4h^2} + \frac{\partial^2 h}{2h^3} - \frac{5(\partial_s h)^2}{4h^4}.
\]

(1.17)

We assume also throughout all this subsection that the following hypothesis holds:

**Assumption 6:**

i) \(k' \in L^\infty(\mathbb{R}), k'' \in L^\infty(\mathbb{R})\)

ii) \(\theta \in C^5(\mathbb{R}), \theta' = \tau \in L^\infty(\mathbb{R}), \theta'' \in L^\infty(\mathbb{R})\).

**Remarks:** As for the 2-dimensional case, note that such operator \(H_k\) (defined by (1.12)-(1.17)) admits bound states and that the minimum eigenvalue \(\lambda_1\) is simple and associated with a positive eigenfunction \(\phi_1\) (see [2], [7]). Still note that \((\lambda, \phi)\) is an eigenpair of the operator \(H_k\) acting on \(L^2(\Omega_0, dsdu)\) means that \((\lambda, W_g^{-1}\phi)\) is an eigenpair of \(-\Delta^D_\Omega\) acting on \(L^2(\Omega)\) (with \(W_g\) defined by (1.14)).

As for the 2-dimensional case, we obtain two kinds of results. Firstly, under

**Assumption 7:**

i) \(k \in C^5(\mathbb{R}), k^{(i)} \in L^\infty(\mathbb{R})\) for all \(i = 0, \ldots, 5\)

ii) \(\theta \in C^5(\mathbb{R}), \theta^{(i)} \in L^\infty(\mathbb{R})\) for all \(i = 1, \ldots, 5\)

where \(k^{(i)}\) (resp. \(\theta^{(i)}\)) denotes the \(i\)-th derivative of \(k\) (resp. of \(\theta\)), we obtain the following result:

**Theorem 1.3.** Let \(\Omega\) be the curved guide in \(\mathbb{R}^3\) defined as above. Let \(k\) be the first curvature function of \(\Omega\) and let \(\tau\) be the second curvature function (i.e. the torsion) of \(\Omega\). Denote by \(\theta\) be an integral of \(\tau\). Assume that Assumptions 5 to 7 are satisfied. Let \(H_k\) be the operator defined by (1.12)-(1.17) and \((\lambda, \phi)\) be an eigenpair of \(H_k\).

Then \(k^2(s) = -4\frac{\Delta \phi(s, 0, 0)}{\phi(s, 0, 0)^2} - \lambda\) for all \(s\) when \(\phi(s, 0, 0) \neq 0\).
**Remarks:** Recall that in \( \mathbb{R}^3 \), \( k \) is a nonnegative function and that the condition imposed on \( \phi \) (\( \phi(s,0,0) \neq 0 \)) in Theorem 1.3 is satisfied for the positive eigenfunction \( \phi_1 \).

We can restrain the hypotheses upon the regularity of \( k \) and \( \theta \). Under

**Assumption 8:**

i) \( k \in C^3(\mathbb{R}), k^{(i)} \in L^\infty(\mathbb{R}) \) for all \( i = 0, \ldots, 3 \)

ii) \( \theta \in C^3(\mathbb{R}), \theta^{(i)} \in L^\infty(\mathbb{R}) \) for all \( i = 0, \ldots, 3 \)

for a guide with a known torsion, we obtain the following result:

**Theorem 1.4.** Let \( \Omega \) be the curved guide in \( \mathbb{R}^3 \) defined as above. Let \( k \) be the first curvature function of \( \Omega \) and let \( \tau \) be the second curvature function (i.e. the torsion) of \( \Omega \). Denote by \( \theta \) be an integral of \( \tau \) and suppose that \( 0 \leq \theta \leq \frac{\pi}{2} \). Assume that Assumptions 5,6 and 8 are satisfied. Let \( H_k \) be the operator defined by (1.12)-(1.17) and \((\lambda, \phi)\) be an eigenpair of \( H_k \).

Then the data \((\lambda, \phi, \theta)\) determines uniquely \( k \).

## 2 Proofs of Theorems 1.1 and 1.2

(the 2-dimensional case)

### 2.1 Proof of Theorem 1.1

First, we recall from [1, Remark 25 p.182] the following lemma.

**Lemma 2.1.** For a second-order elliptic operator defined in a domain \( \omega \in \mathbb{R}^n \), if \( \phi \in H^1_0(\omega) \) satisfies

\[
\int_\omega \sum_{i,j} a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} = \int_\omega f \psi \text{ for all } \psi \in H^1_0(\omega)
\]

then

\[(f \in L^2(\omega), a_{ij} \in C^1(\bar{\omega}), D^\alpha a_{ij} \in L^\infty(\omega) \text{ for all } i,j \text{ and for all } \alpha, |\alpha| \leq 1) \]

imply \( \phi \in H^2(\omega) \)

and for \( m \geq 1 \),

\[(f \in H^m(\omega), a_{ij} \in C^{m+1}(\bar{\omega}), D^\alpha a_{ij} \in L^\infty(\omega) \text{ for all } i,j \text{ and for all } \alpha, |\alpha| \leq m + 1) \]

imply \( \phi \in H^{m+2}(\omega) \).
Now we can prove the Theorem 1.1.

We have \( H_\gamma \phi = \lambda \phi \), so \( \phi \in H^1_0(\Omega_0) \) and

\[
\int_{\Omega_0} [c_\gamma (\partial_s \phi)(\partial_s \psi) + (\partial_u \phi)(\partial_u \psi)] = \int_{\Omega_0} [\lambda \phi - V_\gamma \phi] \psi \text{ for all } \psi \in H^1_0(\Omega_0) \tag{2.1}
\]

with \( c_\gamma \) defined by (1.9) and \( V_\gamma \) defined by (1.10).

Step 1. Using Assumption 2, since \( \gamma^{(k)} \in L^\infty(\Omega_0) \) for \( k = 0, 1, 2 \) then \( V_\gamma \in L^\infty(\Omega_0) \) and \( \lambda \phi - V_\gamma \phi \in L^2(\Omega_0) \). From the hypotheses \( \gamma \in C^1(\mathbb{R}) \) and \( \gamma' \in L^\infty(\mathbb{R}) \), we get that \( \gamma^{(k)} \in L^\infty(\Omega_0) \) and \( \gamma^{(k)} \in L^\infty(\Omega_0) \) for any \( \alpha, |\alpha| \leq 1 \), and so, using Lemma 2.1 for the equation (2.1), we obtain that \( \phi \in H^2(\Omega_0) \).

By the same way, we get that \( \lambda \phi - V_\gamma \phi \in H^1(\Omega_0) \), \( c_\gamma \in C^2(\Omega_0) \), \( D^{\alpha} c_\gamma \in L^\infty(\Omega_0) \) for any \( \alpha, |\alpha| \leq 2 \) (from \( \gamma \in C^3(\mathbb{R}) \), \( \gamma^{(k)} \in L^\infty(\mathbb{R}) \) for any \( k = 0, \ldots, 3 \)). Using Lemma 2.1, we obtain that \( \phi \in H^3(\Omega_0) \).

We apply again the Lemma 2.1 to get that \( \phi \in H^4(\Omega_0) \) (since \( \lambda \phi - V_\gamma \phi \in H^2(\Omega_0) \), \( c_\gamma \in C^3(\Omega_0) \), \( D^{\alpha} c_\gamma \in L^\infty(\Omega_0) \) for all \( \alpha, |\alpha| \leq 3 \), from the hypotheses \( \gamma \in C^4(\mathbb{R}) \) and \( \gamma^{(k)} \in L^\infty(\mathbb{R}) \) for \( k = 0, \ldots, 4 \).)

Finally, using Assumption 2 and Lemma 2.1, we obtain that \( \phi \in H^5(\Omega_0) \). Due to the regularity of \( \Omega_0 \) (see [1; Note p.169]), we have \( \phi \in H^5(\mathbb{R}^2) \) and \( \Delta \phi \in H^3(\mathbb{R}^2) \). Since \( \nabla (\Delta \phi) \in H^2(\mathbb{R}^2) \) and \( H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) (see for example [1; Corollary IX.13 p.168]), we can deduce that \( \Delta \phi \) is continuous (see [1; Remark 8 p.154]).

Step 2. Recall that \( H_\gamma \phi = \lambda \phi \). So by (1.8) to (1.10), we can write:

\[
\frac{u^{\gamma'(s)}}{(1-u^{\gamma(s)})} \partial_s \phi(s,u) - \frac{1}{1-u^{\gamma(s)}} \partial_u^2 \phi(s,u) - \partial_u \phi(s,u) + V_s(s,u) \phi(s,u) = \lambda \phi(s,u) \text{ a.e. in } \Omega_0.
\]

Therefore, for \( u = 0 \), we get: \( -\Delta \phi(s,0) - \frac{\gamma^2(s)}{2} \phi(s,0) + V_s(s,0) \phi(s,0) = \lambda \phi(s,0) \) and equivalently,

\[
\gamma^2(s) = -4 \frac{\Delta \phi(s,0)}{\phi(s,0)} - 4\lambda \text{ if } \phi(s,0) \neq 0.
\]

**2.2 Proof of Theorem 1.2**

As in Theorem 1.1, Step 1, from Assumption 3, we obtain that \( \phi \in H^3(\Omega_0) \). Due to the regularity of \( \Omega_0 \) (see [1; Note p.169]), we have \( \phi \in H^3(\mathbb{R}^2) \). Thus \( \nabla \phi \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) (see for example [1; Corollary IX.13 p.168]). Therefore note that \( \phi \) and \( \partial_s \phi \) are bounded.

We prove now that \( (\lambda, \phi) \) determines uniquely \( \gamma \) when \( \gamma \) is a nonnegative function. For that, assume that \( \Omega_1 \) and \( \Omega_2 \) are two quantum guides in \( \mathbb{R}^2 \) with same width \( d \). We denote by \( \gamma_1 \) and \( \gamma_2 \) the curvatures respectively associated with \( \Omega_1 \) and \( \Omega_2 \) and we suppose that each \( \gamma_i \) satisfies Assumption 3 and is a nonnegative function. Assume that \( H_{\gamma_i} \phi = \lambda \phi = H_{\gamma_2} \phi \).

Then \( \phi \) satisfies

\[
-\partial_s (c_{\gamma_1}(s,u) - c_{\gamma_2}(s,u)) \partial_s \phi(s,u) + (V_{\gamma_1}(s,u) - V_{\gamma_2}(s,u)) \phi(s,u) = 0. \tag{2.2}
\]
Assume that $\gamma_1 \neq \gamma_2$.

Step 1. First, we consider the case where (for example) $\gamma_1(s) < \gamma_2(s)$ for all $s \in \mathbb{R}$.

Let $\epsilon > 0$, $\omega := \mathbb{R} \times I_{s}$ with $I_{s} =] - \epsilon, 0 [\$ and $\omega_{R} =] - R, R [\times I_{s}$ with $R > 0$.

Multiplying (2.2) by $\phi$ and integrating over $\omega_{\epsilon}$, we get:

$$
\int_{\omega_{\epsilon}} (c_{\gamma_1} - c_{\gamma_2}) (\partial_{s} \phi)^2 + \int_{\partial \omega_{\epsilon}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s} + \int_{\omega_{\epsilon}} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0. \tag{2.3}
$$

Since $\epsilon << 1$, $V_{\gamma_i}(s, u) \simeq \frac{\gamma_i^2(s)}{\epsilon}$ for $i = 1, 2$, and so $V_{\gamma_1}(s, u) - V_{\gamma_2}(s, u) > 0$ in $\omega_{\epsilon}$.

Moreover, since

$$
c_{\gamma_1}(s, u) - c_{\gamma_2}(s, u) = \frac{u(\gamma_1(s) - \gamma_2(s))(2 - u(\gamma_1(s) + \gamma_2(s)))}{(1 - u\gamma_1(s))^2(1 - u\gamma_2(s))^2}, \tag{2.4}
$$

we have $c_{\gamma_1}(s, u) > c_{\gamma_2}(s, u)$ in $\omega_{\epsilon}$.

Therefore

$$
\int_{\omega_{\epsilon}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)^2 + \int_{\partial \omega_{\epsilon}} (V_{\gamma_1} - V_{\gamma_2})\phi^2 \geq 0.
$$

Note also that

$$
\int_{\partial \omega_{\epsilon}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s} = \lim_{R \to +\infty} \int_{\partial \omega_{R}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s}.
$$

Since the functions $\phi$ and $\partial_{s}\phi$ are bounded, we get:

$$
\int_{\partial \omega_{R}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s} \leq \int_{\partial \omega_{R}} \text{const}(c_{\gamma_1} - c_{\gamma_2})\nu_{s}
$$

$$
\leq \text{const}\int_{-\epsilon}^{0} (c_{\gamma_1}(R, u) - c_{\gamma_2}(R, u)) du + \int_{-\epsilon}^{0} (c_{\gamma_1}(-R, u) - c_{\gamma_2}(-R, u)) du
$$

We have for all $u \in I_{s}$, $c_{\gamma_1}(R, u) - c_{\gamma_2}(R, u) \to 0$ as $R \to +\infty$.

Moreover: $0 \leq c_{\gamma_1}(R, u) - c_{\gamma_2}(R, u) \leq \text{const}(-u) := l(u)$ with $l \in L^{1}(I_{s})$. By the Lebesgue Dominated Convergence Theorem, we obtain that

$$
\int_{\partial \omega_{R}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s} \to 0 \text{ as } R \to +\infty
$$

and so

$$
\int_{\partial \omega_{s}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)\phi\nu_{s} = 0. \tag{2.5}
$$

Thus from (2.3) and (2.5), we get

$$
\int_{\omega_{\epsilon}} (c_{\gamma_1} - c_{\gamma_2})(\partial_{s} \phi)^2 + \int_{\omega_{\epsilon}} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0 \tag{2.6}
$$

with $c_{\gamma_1} - c_{\gamma_2} > 0$ in $\omega_{\epsilon}$ and $V_{\gamma_1} - V_{\gamma_2} > 0$ in $\omega_{\epsilon}$. We can deduce that $\phi = 0$ in $\omega_{\epsilon}$.

Using a unique continuation theorem (see [7; Theorem XIII.63 p.240]), from $H_{v}\phi = \lambda\phi$, noting that $-\Delta(U_{g}^{-1}\phi) = \lambda U_{g}^{-1}\phi = \lambda g^{-1/4}\phi$, (recall that $U_{g}$ is defined by (1.6)) and so $\mid\Delta(U_{g}^{-1}\phi)\mid \leq \mid\lambda g^{-1/4}\phi\mid$ with $g > 0$ a.e., we can deduce that $\phi = 0$ in $\Omega_{0}$. So we get a contradiction (since $\phi$ is an eigenfunction).

Step 2. From Step 1, we obtain that there exists at least one point $s_{0} \in \mathbb{R}$ such that $\gamma_1(s_{0}) = \gamma_2(s_{0})$. Since $\gamma_1 \neq \gamma_2$, we can choose $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that
(for example) \( \gamma_1(a) = \gamma_2(a) \), \( \gamma_1(s) < \gamma_2(s) \) for all \( s \in ]a, b[ \) and \( \gamma_1(b) = \gamma_2(b) \) if \( b \in \mathbb{R} \).

We proceed as in Step 1, considering, in this case, \( \omega_\epsilon := ]a, b[ \times I_\epsilon \). We study again the equation (2.3) and as in Step 1, we prove that

\[
\int_{\partial \omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s \phi) \phi \nu_s = 0.
\]

Indeed from (2.4) and \( \gamma_1(a) = \gamma_2(a) \) we have \( c_{\gamma_1}(a, u) = c_{\gamma_2}(a, u) \). If \( b \in \mathbb{R} \), we have also \( c_{\gamma_1}(b, u) = c_{\gamma_2}(b, u) \) and otherwise if \( b = +\infty \), we use again the Lebesgue Dominated Convergence Theorem to obtain that \( \int_{\partial \omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s \phi) \phi \nu_s = 0 \). So the equation (2.3) becomes

\[
\int_{\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s \phi)^2 + \int_{\omega_\epsilon} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0
\]

with \( c_{\gamma_1} - c_{\gamma_2} > 0 \) in \( \omega_\epsilon \) and \( V_{\gamma_1} - V_{\gamma_2} > 0 \) in \( \omega_\epsilon \). So \( \phi = 0 \) in \( \omega_\epsilon \) and as in Step 1, by a unique continuation theorem, we obtain that \( \phi = 0 \) in \( \Omega_0 \). Therefore we get a contradiction.

**Remarks:**

1. Note that the previous theorem is true if we replace the hypothesis ”\( \gamma \) is non-negative” by the hypothesis ”\( \gamma \) is nonpositive”. Indeed, in this last case, we just have to take \( I_\epsilon = ]0, \epsilon[ \) and the proof rests valid.

2. If \( \gamma \) satisfies Assumption 4 and if \( \gamma \) is a nonnegative function (for example) with compact support, the Theorem 1.2 is still valid. Indeed, if we take again \( \Omega_1 \) and \( \Omega_2 \) two quantum guides in \( \mathbb{R}^2 \) and if we still use the same notations precisied in the proof of Theorem 1.2, we can adapt the above proof for this particular case.

   We still assume that \( H_{\gamma_1} \phi = \lambda \phi = H_{\gamma_2} \phi \).

   Then \( \phi \) satisfies the equation (2.3) and here again we want to prove that \( \gamma_1 = \gamma_2 \).

   Assume that \( \gamma_1 \neq \gamma_2 \). Since each curvature \( \gamma_i \) has a compact support, there exists two points \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \) such that \( \gamma_1(a) = \gamma_2(a) \), \( \gamma_1(b) = \gamma_2(b) \) and \( \gamma_1(s) < \gamma_2(s) \) for all \( s \in ]a, b[ \) (for example). For this case we denote by \( \omega_\epsilon = ]a, b[ \times I_\epsilon \) and we proceed exactly as in the proof of Theorem 1.2: due to the choice of \( \omega_\epsilon \), (2.5) is verified and thus we still obtain (2.6). We use again a unique continuation theorem to get a contradiction. Note that here we do not need to prove that \( \phi \) and \( \partial_s \phi \) are bounded.

3 Proofs of Theorem 1.3 and 1.4

**(the 3-dimensional case)**
3.1 Proof of Theorem 1.3

We proceed exactly as for Theorem 1.1. The proof is based on Lemma 2.1. We have $H_k\phi = \lambda \phi$ with $\phi \in H^1_0(\Omega_0)$. So

$$
\int_{\Omega_0} [h^{-2}(\partial_s \phi)(\partial_s \psi) + (\partial_{u_2} \phi)(\partial_{u_2} \psi) + (\partial_{u_3} \phi)(\partial_{u_3} \psi)] = \int_{\Omega_0} [\lambda \phi - V_k \phi] \psi \quad \text{for all } \psi \in H^1_0(\Omega_0)
$$

(3.1)

with $h$ defined by (1.12) and $V_k$ defined by (1.17).

Step 1. From Assumptions 5 and 6, since $k, k', k'', \theta', \theta''$ are bounded, we deduce that $V_k \in L^\infty(\Omega_0)$. Therefore $\lambda \phi - V_k \phi \in L^2(\Omega_0)$ and $\partial^\alpha (h^{-2}) \in L^\infty(\Omega_0)$ for any $\alpha, |\alpha| \leq 1$. Thus, using Lemma 2.1 for the equation (3.1), we obtain that $\phi \in H^2(\Omega_0)$.

By the same way, we get that $\lambda \phi - V_k \phi \in H^1(\Omega_0)$, $h^{-2} \in C(\Omega_0)$ and $\partial^\alpha (h^{-2}) \in L^\infty(\Omega_0)$ for any $\alpha, |\alpha| \leq 2$ (since $k \in C^3(\mathbb{R})$, $\theta \in C^3(\mathbb{R})$ and all of their derivatives are bounded). Using Lemma 2.1, we obtain that $\phi \in H^3(\Omega_0)$.

From the hypotheses $k \in C^4(\mathbb{R})$, $\theta \in C^4(\mathbb{R})$, $k(i) \in L^\infty(\mathbb{R})$ for all $i = 0, \ldots, 4$ and $\theta(i) \in L^\infty(\mathbb{R})$ for all $i = 1, \ldots, 4$, using Lemma 2.1, we obtain that $\phi \in H^4(\Omega_0)$.

Finally, using Assumption 7 and Lemma 2.1, we obtain that $\phi \in H^5(\Omega_0)$. Due to the regularity of $\Omega_0$, we have $\phi \in H^5(\mathbb{R}^3)$ and $\Delta \phi \in H^3(\mathbb{R}^3)$. Since $\nabla (\Delta \phi) \in H^2(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ (see for example [1; Corollary IX.13 p.168]), we can deduce that $\Delta \phi$ is continuous (see [1; Remark 8 p.154]).

Step 2. Recall that $H_k \phi = \lambda \phi$. Therefore, for $u = (u_2, u_3) = (0, 0)$, we get:

$$
-\Delta \phi(s, 0, 0) - \frac{k^2(s)}{4} \phi(s, 0, 0) = \lambda \phi(s, 0, 0)
$$

and equivalently, $k^2(s) = -4\frac{\Delta \phi(s, 0, 0)}{\phi(s, 0, 0)} - 4\lambda$ if $\phi(s, 0, 0) \neq 0$.

3.2 Proof of Theorem 1.4

As in Theorem 1.3, by Assumption 8, we obtain that $\phi \in H^3(\Omega_0)$. Due to the regularity of $\Omega_0$, we have $\phi \in H^3(\mathbb{R}^3)$. Thus $\nabla \phi \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R}^3)$. Therefore note that $\phi$ and $\partial_s \phi$ are bounded.

We prove now that $(\lambda, \phi, \theta)$ determines uniquely $k$.

Assume that $\Omega_1$ and $\Omega_2$ are two guides in $\mathbb{R}^3$. We denote by $k_1$ and $k_2$ the first curvatures functions associated with $\Omega_1$ and $\Omega_2$ and we denote by $\theta$ an integral of $\tau$ the common torsion of $\Omega_1$ and $\Omega_2$. We suppose that $k_1, k_2$ and $\theta$ satisfy the Assumption 8 and that $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$. Assume that $H_{k_1} \phi = \lambda \phi = H_{k_2} \phi$.

Then $\phi$ satisfies

$$
-\partial_s ((h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3))\partial_s \phi(s, u_2, u_3)) + (V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3)) \phi(s, u_2, u_3) = 0
$$

(3.2)

where $h_1$ (associated with $k_1$) is defined by (1.12), $V_{k_1}$ is defined by (1.17), $h_2$ (associated with $k_2$) is defined by (1.12) and $V_{k_2}$ is defined by (1.17).
Assume that \( k_1 \neq k_2 \).

Step 1. First, we consider the case where (for example) \( k_1(s) < k_2(s) \) for all \( s \in \mathbb{R} \). Recall that each \( k_i \) is a nonnegative function.

Let \( \epsilon > 0 \) and denote by \( J_\epsilon := ]-\epsilon, [\epsilon[ - \epsilon, 0[ \times J_\epsilon \) and \( O_\epsilon := \mathbb{R} \times J_\epsilon \) and \( O_R := ]-R, R[\times J_\epsilon \)

with \( \epsilon \) small enough to have \( J_\epsilon \subset \omega \) and \( R > 0 \) (recall that \( \Omega_0 = \mathbb{R} \times \omega \)).

Multiplying (3.2) by \( \phi \) and integrating over \( O_\epsilon \), we get:

\[
\int_{O_\epsilon} (h_1^{-2} - h_2^{-2}) (\partial_s \phi)^2 + \int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s + \int_{O_\epsilon} (V_{k_1} - V_{k_2}) \phi^2 = 0. \tag{3.3}
\]

Since \( \epsilon < 1, V_{k_i} \simeq -\frac{k_i^2(s)}{4} \) for \( i = 1, 2 \), and so \( V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3) > 0 \) in \( O_\epsilon \).

Moreover, note that:

\[
h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3) = \frac{\alpha(s, u_2, u_3)(k_1(s) - k_2(s))(h_1(s, u_2, u_3) + h_2(s, u_2, u_3))}{h_1^2(s, u_2, u_3)h_2^2(s, u_2, u_3)} \tag{3.4}
\]

with \( \alpha(s, u_2, u_3) := \cos(\theta(s))u_2 + \sin(\theta(s))u_3 \).

Since \((u_2, u_3) \in J_\epsilon \) and \( 0 \leq \theta(s) \leq \frac{\pi}{2} \) for all \( s \in \mathbb{R} \), we have \( \alpha(s, u_2, u_3) < 0 \). Therefore, by (3.4), we deduce that \( h_1^{-2} - h_2^{-2} > 0 \) in \( O_\epsilon \).

Thus \( \int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{O_\epsilon} (V_{k_1} - V_{k_2}) \phi^2 \geq 0 \).

Note also that: \( \int_{O_\epsilon} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s = \lim_{R \to +\infty} \int_{0O_R} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s = \lim_{R \to +\infty} \int_{\partial O_R} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s \).

Since the functions \( \phi \) and \( \partial_s \phi \) are bounded, we get:

\[
| \int_{\partial O_R} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s | \leq \text{const} \int_{\partial O_R} |(h_1^{-2} - h_2^{-2}) \nu_s |
\]

\[
\leq \text{const} \left[ \int_{J_\epsilon} (h_1^{-2} (-R, u_2, u_3) - h_2^{-2} (-R, u_2, u_3)) du_2 du_3 \right.
\]

\[
+ \int_{J_\epsilon} (h_1^{-2} (R, u_2, u_3) - h_2^{-2} (R, u_2, u_3)) du_2 du_3 \right].
\]

We have for all \( u = (u_2, u_3) \in J_\epsilon, (h_1^{-2} - h_2^{-2})(R, u_2, u_3) \to 0 \) as \( R \to \infty \).

Moreover: \( 0 \leq h_1^{-2}(R, r, \theta) - h_2^{-2}(R, u_2, u_3) \leq A u_2 + B u_3 \in L^1(J_\epsilon) \) with \( A \) and \( B \) constants.

By the Lebesgue Dominated Convergence Theorem, we obtain that

\[
\int_{J_\epsilon} (h_1^{-2} (R, u_2, u_3) - h_2^{-2} (R, u_2, u_3)) du_2 du_3 \to 0 \text{ as } R \to +\infty
\]

so

\[
\int_{\partial O_R} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s \to 0 \text{ as } R \to +\infty.
\]

Thus

\[
\int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2}) (\partial_s \phi) \phi \nu_s = 0. \tag{3.5}
\]
Therefore, from (3.3) and (3.5) we get:

\[
\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 = 0
\]  

(3.6)

with \( h_1^{-2} - h_2^{-2} > 0 \) in \( O_\epsilon \) and \( V_{k_1} - V_{k_2} > 0 \) in \( O_\epsilon \).

From (3.6) we can deduce that \( \phi = 0 \) in \( O_\epsilon \).

Using a unique continuation theorem (see [7; Theorem XIII.63 p.240]), from \( H_{k_1} \phi = \lambda \phi \), noting that \(-\Delta(W^{-1}_g\phi) = \lambda W^{-1}_g\phi = \lambda h_1^{-1/2}\phi \) with \( h_1 > 0 \) a.e., we can deduce that \( \phi = 0 \) in \( \Omega_0 \). So we get a contradiction.

Step 2. From Step 1, we obtain that there exists at least one point \( s_0 \in \mathbb{R} \) such that \( k_1(s_0) = k_2(s_0) \). Since \( k_1 \neq k_2 \), we can choose \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \cup \{+\infty\} \) such that (for example) \( k_1(a) = k_2(a) \), \( k_1(s) < k_2(s) \) for all \( s \in [a, b] \) and \( k_1(b) = k_2(b) \) if \( b \in \mathbb{R} \).

We proceed as in Step 1, considering in this case \( O_\epsilon := [a, b] \times J_\epsilon \).

From \( k_1(a) = k_2(a) \), we get that \( h_1^{-2}(a, u_2, u_3) = h_2^{-2}(a, u_2, u_3) \).

If \( b \in \mathbb{R} \), we have also \( h_1^{-2}(b, u_2, u_3) = h_2^{-2}(b, u_2, u_3) \) and otherwise, if \( b = +\infty \), we use again the Lebesgue Dominated Convergence Theorem to obtain that \( \int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)\phi \nu_s = 0 \). So (3.3) becomes

\[
\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 = 0 \quad \text{with} \quad h_1^{-2} - h_2^{-2} > 0 \quad \text{in} \quad O_\epsilon \quad \text{and} \quad V_{k_1} - V_{k_2} > 0 \quad \text{in} \quad O_\epsilon .
\]

So \( \phi = 0 \) in \( O_\epsilon \), and as in Step 1, by a unique continuation theorem, we obtain that \( \phi = 0 \) in \( \Omega_0 \). Therefore we get a contradiction.

4 Generalization in \( \mathbb{R}^n \)

References